Definitions
Geometry qualifying course
MSU, Fall 2016

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This document was made as a way to study the material from the fall semester differential geometry qualifying course at Michigan State University, in fall of 2016. It serves as a companion document to the "Definitions" review sheet for the same class. The main textbook for the course was *Introduction to Smooth Manifolds* by John Lee, and this document closely follows the order of material in that book.

Contents

0 Appendix A: Topology

Theorem 0.1. *Let $f : X \to Y$ be a continuous map between topological spaces. If $K \subset X$ is compact, then $f(K) \subset Y$ is compact.*

1 Chapter 1 - Defining manifolds

Theorem 1.1 (Topological Invariance of Dimension)**.** *A nonempty n-dimensional topological manifold cannot be homeomorphic to an m-dimensional manifold unless $m = n$.*

Theorem 1.2. *Every topological manifold has a countable basis of precompact open balls.*

Theorem 1.3. *Let $M_1, \ldots, M_k$ be topological manifolds of dimensions $n_1, \ldots, n_k$ respectively. The product space $\prod_i M_i$ is a topological manifold of dimension $\sum_i n_i$. For $(p_i) \in \prod_i M_i$, we choose charts (U_i, ϕ_i) in each M_i with $p_i \in U_i$ and then the maps*

$$\phi_1 \times \ldots \times \phi_k : \prod_i U_i \to \mathbb{R}^{n_1 + \ldots + n_k}$$

given by

$$(p_1, \ldots, p_k) \mapsto (\phi_1(p_1), \ldots, \phi_k(p_k))$$

are the coordinate charts.

Theorem 1.4. *Every topological manifold is locally compact.*

Theorem 1.5. *Every topological manifold is paracompact.*

Theorem 1.6. *A topological manifold has countably many components, each of which is an open subset and a connected topological manifold.*

Theorem 1.7. *Every smooth atlas for a manifold M is contained in a unique maximal smooth atlas.*

Theorem 1.8. *Two smooth atlases for a manifold M determine the same smooth structure if and only if their union is a smooth atlas.*

Theorem 1.9. *Every smooth manifold has a countable basis of regular coordinate balls.*

Theorem 1.10. *Any open subset U of a manifold M is also a manifold, by forming smooth charts for U by taking intersections of U with smooth charts for M. (It has the same dimension provided U is nonempty.)*

Theorem 1.11. *Let $M_1, \ldots, M_k$ be smooth manifolds of dimension $n_1, \ldots, n_k$ respectively. Then the charts defined for the product manifold are smoothly compatible.*

Theorem 1.12 (Smooth Manifold Chart Lemma)**.** *Let M be a set and suppose there is a collection $\{U_\alpha\}$ of subsets and a collection of maps $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ so that*

1. *ϕ_α is a bijection with open image in $\mathbb{R}^n$*

2. $\phi_\alpha(U_\alpha \cap U_\beta)$ *is open for every* α, β

3. *If* $U_\alpha \cap U_\beta \neq \emptyset$, *the map* $\phi_\beta \circ \phi_\alpha^{-1}$ *is smooth*

4. *A countable subcollection of* U_α *is a cover for* M

5. *For* $p, q \in M$, *either both are contained in some* U_α *or there are disjoint* U_α, U_β *so that* $p \in U_\alpha$ *and* $q \in U_\beta$.

Then M *has a unique smooth manifold structure such that* (U_α, ϕ_α) *are smooth charts.*

2 Chapter 2 - Smooth functions

Theorem 2.1. *Let* M *be a smooth manifold and* $f : M \to \mathbb{R}$. *Then* f *is smooth if and only if* $\widehat{f}$ *is smooth in every smooth chart.*

Theorem 2.2. *Let* M *be a smooth manifold. Then* $C^\infty(M)$ *is a commutative ring (with pointwise addition and multiplication).*

Theorem 2.3. *Smooth maps are continuous.*

Theorem 2.4. *Let* M, N *be manifolds and* $F : M \to N$ *be a map. The following are equivalent:*

1. F *is smooth.*

2. *For every* $p \in M$, *there exist smooth charts* $(U, \phi), (V, \psi)$ *with* $p \in U$, $F(p) \in V$ *such that* $U \cap F^{-1}(V)$ *is open in* M *and* $\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(V)$ *is smooth.*

3. F *is continuous and there exist smooth atlases* $\{U_\alpha, \phi_\alpha\}$ *for* M *and* $\{V_\beta, \psi_\beta\}$ *for* N *such that* $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ *is smooth for all* α, β.

4. *For every* $p \in M$, *there is a neighborhood* U *such that* $F|_U$ *is smooth.*

Theorem 2.5 (Gluing Lemma for Smooth Maps)**.** *Let* M, N *be smooth manifolds, and let* $\{U_\alpha\}_{\alpha \in A}$ *be an open cover for* M. *Suppose that for each* $\alpha \in A$ *we have a smooth map* $F_\alpha : U_\alpha \to N$ *such that they agee on overlaps, that is,*

$$F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$$

for all α, β. *Then there is a unique smooth map* $F : M \to N$ *such that* $F|_{U_\alpha} = F_\alpha$ *for all* $\alpha \in A$.

Theorem 2.6. *Let* M, N *be smooth manifolds, and let* $F : M \to N$ *be a map. Then* F *is smooth if and only if every coordinate representation of* F *is smooth.*

Theorem 2.7. *Constant maps are smooth.*

Theorem 2.8. *The identity map is smooth.*

Theorem 2.9. *Inclusion maps are smooth.*

Theorem 2.10. *The composition of smooth maps is smooth.*

Theorem 2.11. *Let $M_1, \dots, M_k$ be smooth manifolds. Let $\pi_i : \prod_{j=1}^k M_j \to M_i$ be the projection $(p_1, \dots, p_k) \mapsto p_i$. A map $F = (F_1, \dots F_k) : N \to \prod_{i=1}^k M_i$ is smooth if and only if each $F_i = \pi_i \circ F : N \to M_i$ is smooth.*

Theorem 2.12. *Let M be a smooth manifold and (U, ϕ) be a smooth chart. Then $\phi : U \to \phi(U)$ is a diffeomorphism.*

Theorem 2.13. *The composition of diffeomorphisms is a diffeomorphism.*

Theorem 2.14. *Diffeomorphism is an equivalence relation on smooth manifolds.*

Theorem 2.15 (Diffeomorphism Invariance of Dimension)**.** *Smooth manifolds can only be diffeomorphic if they have the same dimension, unless one is empty (in which case both are empty).*

Theorem 2.16. *Let $f : M \to \mathbb{R}^k$. If $x \in M \setminus \operatorname{supp}(f)$, then $f(x) = 0$. Note that we may have $f(y) = 0$ for $y \in \operatorname{supp}(f)$.*

Proof. If $f(x) \neq 0$, then $x \in \operatorname{supp}(f)$. □

Theorem 2.17. *Let M be a smooth manifold and $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ an open cover. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.*

Theorem 2.18. *Let M be a smooth manifold and A, U be subsets with $A \subset U$ and A closed and U open. Then there exists a smooth bump function for A supported in U.*

Theorem 2.19. *Let M be a smooth manifold, with $A \subset M$ closed and $f : A \to \mathbb{R}^k$ a smooth function. Then for any open subset U with $A \subset U$, there exists a smooth function $\widetilde{f} : M \to \mathbb{R}^k$ such that $\widetilde{f}|_A = f$ and $\operatorname{supp} \widetilde{F} \subset U$.*

Theorem 2.20. *Let M be a smooth manifold. Then there exists a smooth exhaustion function for M, that is, there exists a smooth function $f : M \to \mathbb{R}$ such that $f^{-1}((-\infty, a])$ is compact for all $a \in \mathbb{R}$.*

Theorem 2.21. *Let M be a smooth manifold and K a closed subset of M. Then there is a smooth nonnegative function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$.*

3 Chapter 3 - Tangent bundle

Theorem 3.1. *Let M be a smooth manifold and $p \in M$. Then T_pM is a vector space (over $\mathbb{R}$).*

Proof. Let $v, w \in T_pM$ and $a \in \mathbb{R}$. We need to show that $v + w \in T_pM$ and $av \in T_pM$, so we need to show that $v + w$ and av are derivations at p.

$$\begin{aligned}(v+w)(fg) &= v(fg) + w(fg) = f(p)vg + g(p)vf + f(p)wg + g(p)wf \\ &= f(p)(vg + wg) + g(p)(vf + wf) = f(p)(v+w)g + g(p)(v+w)f\end{aligned}$$

Thus $v + w$ is a derivation at p.

$$\begin{aligned}(av)(fg) &= a(v(fg)) = a(f(p)vg + g(p)vf) = f(p)(avg) + g(p)(avf) \\ &= g(p)(av)f + f(p)(av)g\end{aligned}$$

Thus av is a derivation at p. □

Theorem 3.2. *Let M be a smooth manifold, and let $p \in M, v \in T_pM$, and $f, g \in C^\infty(M)$. If f is a constant function, then $vf = 0$. If $f(p) = g(p) = 0$, then $v(fg) = 0$.*

Proof. First suppose that f is the constant function $f(p) = 1$.

$$vf = v(ff) = f(p)vf + f(p)vf = 2vf \implies vf = 0$$

Then by linearity, if $g(p) = c$ we have $vg = v(cf) = c(vf) = c(0) = 0$. The other assertion is obvious. □

Theorem 3.3. *Let M, N be smooth manifolds, $F : M \to N$ be smooth, and $p \in M$. Then $dF_p(v) = F_*(v)$ is a derivation at $F(p)$.*

Theorem 3.4. *Let M, N, P be smooth manifolds, and $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$. Then*

1. *$dF_p = F_* : T_pM \to T_{F(p)}N$ is linear.*
2. *$d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$. In the other notation, $(G \circ F)_* = G_* \circ F_*$*
3. *$d(\mathrm{Id}_M)_p = (\mathrm{Id}_M)_* = \mathrm{Id}_{T_pM}$.*
4. *If F is a diffeomorphism, then $dF_p = F_*$ is an isomorphism, and $(F_*)^{-1} = (F^{-1})_*$.*

Theorem 3.5. *Let M be a smooth manifold, $p \in M$, and $v \in T_pM$. Let $f, g \in C^\infty(M)$ and suppose there is a neighborhood U of p such that $f(x) = g(x)$ for $x \in U$. Then $vf = vg$.*

Theorem 3.6. *Let M be a smooth manifold and $U \subset M$ be open. Let $\iota : U \hookrightarrow M$ be the inclusion map. Then for $p \in U$, the differential $d\iota_p = \iota_* : T_pU \to T_pM$ is an isomorphism.*

Theorem 3.7. *Let M be a smooth n-manifold. Then for $p \in M$, T_pM is an n-dimensional vector space (over $\mathbb{R}$).*

Theorem 3.8. *Let V be a finite-dimensional (real) vector space with its standard smooth manifold structure. For $a \in V$ the map $v \to D_v|a$ is a canonical isomorphism from V to T_aV, and for any linear map $L : V \to W$, the diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{\cong} & T_aV \\ \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L_*} \\ W & \xrightarrow{\cong} & T_{La}W \end{array}$$

Theorem 3.9. *Let $M_1, \ldots M_k$ be smooth manifolds and let $\pi_j : (M_1 \times \ldots \times M_k) \to M_j$ be the projection $(p_1, \ldots p_k) \to p_j$. Then*

$$\alpha : T_p(M_1 \times \ldots \times M_k) \to T_{p_1}M_1 \oplus \ldots \oplus T_{p_k}M_k$$
$$\alpha(v) = ((\pi_1)_*(v), \ldots, (\pi_k)_*(v)) = (d(\pi_1)_p(v), \ldots, d(\pi_k)_p(v))$$

is an isomorphism.

Theorem 3.10. *Let $p \in \mathbb{R}^n$. The derivations*

$$\left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p$$

form a basis for $T_p\mathbb{R}^n$.

Theorem 3.11. *Let M be a smooth n-manifold and $p \in M$. Then T_pM is an n-dimensional (real) vector space, and for any smooth chart $(U, \phi) = (U, (x^1, \ldots, x^n))$ containing p, the coordinate vectors*

$$\left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p$$

form a basis for T_pM. Therefore, any $v \in T_pM$ can be written as

$$v = \sum_{i=1}^{n} v^i \left.\frac{\partial}{\partial x^i}\right|_p$$

Furthermore, $x^j : U \to \mathbb{R}$ is a smooth function, so

$$v(x^j) = \left(v^i \left.\frac{\partial}{\partial x^i}\right|_p \right)(x^j) = v^i \frac{\partial x^j}{\partial x^i}(p) = v^j$$

Theorem 3.12. *Let M, N be smooth manifolds and let $F : M \to N$ be a smooth map. Let (U, ϕ) be a chart for M with $p \in U$ and (V, ψ) be a chart for N with $F(p) \in V$. Let $\widehat{F} = \psi \circ F \circ \phi^{-1}$ be the coordinate representation of F and $\widehat{p} = \phi(p)$ be the coordinate representation of p. Then*

$$F_* \left(\left.\frac{\partial}{\partial x^i}\right|_p \right) = \frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p}) \left.\frac{\partial}{\partial y^j}\right|_{F(p)}$$

That is, the matrix of the linear map $dF_p = F_*$ is the Jacobian matrix of F at p,

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \ldots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial x^1}(p) & \ldots & \frac{\partial F^n}{\partial x^n}(p) \end{pmatrix}$$

Theorem 3.13 ("Chain Rule" for Coordinate Changes)**.** *Let M be a smooth n-manifold and $(U,\phi) = (U,x^i)$ and $(V,\psi) = (V,y^i)$ be two smooth charts. The change of basis from $\left\{ \frac{\partial}{\partial x^i}\big|_p \right\}$ to $\left\{ \frac{\partial}{\partial y^i}\Big|_p \right\}$ is given by*

$$\frac{\partial}{\partial x^i}\bigg|_p = \sum_{j=1}^{n} \left(\frac{\partial y^j}{\partial x^i}(\widehat{p}) \right) \frac{\partial}{\partial y^j}\bigg|_p$$

where $\widehat{p} = \phi(p)$. For $v \in T_pM$, we can write v in terms of both bases as

$$v = a^i \, \frac{\partial}{\partial x^i}\bigg|_p = b^i \, \frac{\partial}{\partial y^i}\bigg|_p$$

Then the relationship between a^i and b^i is given by

$$b^i = \sum_{i=1}^{n} \left(\frac{\partial y^j}{\partial x^i}(\widehat{p}) \right) a^i$$

Theorem 3.14. *Let M be a smooth n-manifold. Then TM is a smooth manifold of dimension $2n$.*

Theorem 3.15. *Let M be a smooth n-manifold such that M can be covered by a single smooth chart. Then TM is diffeomorphic to $M \times \mathbb{R}^n$.*

Theorem 3.16. *Let M, N be smooth manifolds and $F : M \to N$ be smooth. Then $dF : TM \to TN$ is smooth.*

Theorem 3.17. *Let M, N, P be smooth manifolds and $F : M \to N, G : N \to P$ be smooth maps. Then*

1. *$d(G \circ F) = dG \circ dF$*
2. *$d(\mathrm{Id}_M) = \mathrm{Id}_{TM}$*
3. *If F is a diffeomorphism, then $dF : TM \to TN$ is also a diffeomorphism and $(dF)^{-1} = d(F^{-1})$.*

Theorem 3.18. *Let M be a smooth manifold and $\gamma : J \to M$ be a smooth curve. Then for a function $f \in C^\infty(M)$, $\gamma'(t_0)$ acts on f by*

$$\gamma'(t_0)f = d\gamma \left(\frac{d}{dt}\bigg|_{t_0} \right) f = \frac{d}{dt}\bigg|_{t_0} (f \circ \gamma) = (f \circ \gamma')(t_0)$$

Theorem 3.19. *Let M be a smooth manifold and $\gamma : J \to M$ be a smooth curve, and let $t_0 \in J$. Let (U,ϕ) be a smooth chart containing t_0, with $\phi(p) = (x^1(p), \ldots x^n(p))$. Then for t sufficiently close to t_0, we can write γ as*

$$\gamma(t) = (\gamma^1(t), \ldots \gamma^n(t))$$

and then

$$\gamma'(t_0) = \frac{d\gamma^i}{dt}(t_0) \, \frac{\partial}{\partial x^i}\bigg|_{\gamma(t_0)}$$

Theorem 3.20. *Let M be a smooth manifold and $p \in M$, and $v \in T_pM$. Then there exists a curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$ for some $\epsilon > 0$.*

Theorem 3.21. *Let $F : M \to N$ be a smooth map and let $\gamma : J \to M$ be a smooth curve. For $t_0 \in J$, the velocity of $F \circ \gamma : J \to N$ at t_0 is*

$$(F \circ \gamma)'(t_0) = dF(\gamma'(t_0))$$

Theorem 3.22. *Let $F : M \to N$ be a smooth map, and $p \in M$ and $v \in T_pM$. Then*

$$dF_p(v) = F_*(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma : J \to M$ satisfying $0 \in J$, $\gamma(0) = p$, and $\gamma'(0) = v$.

4 Chapter 4 - Submersions and immersions

Theorem 4.1. *Let $F : M \to N$. For all $p \in M$, the rank of F at p does not exceed $\min(\dim M, \dim N)$.*

Theorem 4.2. *Let $F : M \to N$ be a smooth map and $p \in M$. If $dF_p = F_*$ is surjective, then there is a neighborhood U of p such that $F|_U$ is a submersion. Similarly, if $dF_p = F_*$ is injective, then there is a neighborhood U of p such that $F|_U$ is an immersion.*

Theorem 4.3. *Let $M_1, \ldots M_k$ be smooth manifolds, and let $\pi : M_1 \times \ldots \times M_k \to M_i$ be the projection $(p_1, \ldots, p_k) \mapsto p_i$. Then π_i is a smooth submersion.*

Theorem 4.4. *Let M be a smooth manifold and $\gamma : J \to M$ a smooth curve. Then γ is a smooth immersion if and only if $\gamma'(t)$ is never zero.*

Theorem 4.5. *Let M be a smooth manifold and TM the tangent bundle. Then the standard projection $\pi : TM \to M$ given by $(p, v) \mapsto p$ is a smooth submersion.*

Theorem 4.6. *A composition of surjective functions is surjective.*

Theorem 4.7. *A composition of injective functions is injective.*

Theorem 4.8. *A composition of smooth submersions is a smooth submersion.*

Theorem 4.9. *A composition of smooth immersions is a smooth immersion.*

Theorem 4.10 (Properties of Local Diffeomorphisms)**.**

1. *A composition of local diffeomorphisms is a local diffeomorphism.*
2. *A finite product of local diffeomorphisms is a local diffeomorphism.*
3. *A bijective local diffeomorphism is a diffeomorphism.*
4. *A local diffeomorphism is a smooth immersion and a smooth submersion. Conversely, a map that is a smooth submersion and smooth immersion is a local diffeomorphism.*

5. *A smooth immersion between manifolds of equal dimension is a local diffeomorphism. Likewise for smooth submersions.*

Theorem 4.11. *Let $F : M \to N$ be a smooth map. Then F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion. Consequently, if $\dim M = \dim N$ and F is either a smooth sumbersion or a smooth immersion, then F is a local diffeomorphism.*

Theorem 4.12 (Inverse Function Theorem)**.** *Let M, N be smooth manifolds and $F : M \to N$ a smooth map. If $p \in M$ such that $dF_p = F_*$ is invertible, then there are connected neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

Theorem 4.13 (Rank Theorem)**.** *Let M, N be smooth manifolds of dimension m, n respectively and $F : M \to N$ a smooth map with constant rank r. Then for $p \in M$ there exist smooth charts (U, ϕ) for M with $p \in U$ and (V, ψ) for N with $F(p) \in V$ such that $F(U) \subset V$ and F has a coordinate representation of the form*

$$\widehat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots x^m) = (x^1, \ldots, x^r, 0, \ldots 0)$$

In particular, if F is a smooth submersion, then

$$\widehat{F}(x^1, \ldots x^n, x^{n+1}, \ldots, x^m) = (x^1, \ldots x^n)$$

and if F is a smooth immersion then

$$\widehat{f}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0)$$

Theorem 4.14. *Let $F : M \to N$ be a smooth map and suppose M is connected. Then the following are equivalent:*

1. *F has constant rank.*

2. *For $p \in M$, there exist smooth charts (U, ϕ) and (V, ψ) with $p \in U$, $F(p) \in V$, such that the coordinate representation $\widehat{F} = \psi \circ F \circ \phi^{-1}$ is linear.*

Theorem 4.15 (Global Rank Theorem)**.** *Let M, N be smooth manifolds such that $F : M \to N$ is a smooth map of constant rank. Then*

1. *If F is surjective, then it is a smooth submersion.*

2. *If F is injective, then it is a smooth immersion.*

3. *If F is bijective, then it is a diffeomorphism.*

Theorem 4.16. *The composition of smooth embeddings is a smooth embedding.*

Theorem 4.17. *If M is a smooth manifold and $U \subset M$ is an open submanifold, then the inclusion map is a smooth embedding.*

Theorem 4.18. *Let $M_1, \dots, M_k$ be smooth manifolds and $p_i \in M_i$ be points. Then the map $\iota_j : M_j \to \prod_i M_i$ given by*

$$\iota_j(q) = (p_1, \dots, p_{j-1}, q, p_{j+1}, \dots, p_k)$$

is a smooth embedding.

Theorem 4.19. *Let M, N be smooth manifolds and $F : M \to N$ be an injective smooth immersion. If any of the following holds, then F is a smooth embedding.*

1. *F is an open or closed map.*

2. *F is a proper map.*

3. *M is compact.*

4. *M has empty boundary and $\dim M = \dim N$.*

Theorem 4.20 (Local Embedding Theorem)**.** *Let M, N be smooth manifolds and $F : M \to N$ be a smooth map. Then F is a smooth immersion if and only if every $p \in M$ has a neighborhood $U \subset M$ such that $F|_U : U \to N$ is a smooth embedding.*

Theorem 4.21 (Local Section Theorem)**.** *Let M, N be smooth manifolds and $\pi : M \to N$ a smooth map. Then π is a smooth submersion if and only if every $p \in M$ is in the image of a local section of π.*

Theorem 4.22. *Let M, N be smooth manifolds and $\pi : M \to N$ a smooth submersion. Then π is an open map. If π is surjective, then it is a quotient map.*

5 Chapter 5 - Critical points of smooth functions

Theorem 5.1. *Let M be a smooth manifold. The embedded submanifolds of codimension zero in M are exactly the open submanifolds.*

Theorem 5.2. *Let M, N be smooth manifolds, and $F : N \to M$ be a smooth embedding. Let $S = F(N)$. Then with the subspace topology (from M), S is a topological manifold, and it has a unique smooth structure making it into an embedded submanifold of M with the property that F is a diffeomorphism between N and $F(N)$.*

Theorem 5.3. *Let M, N be smooth manifolds and let $p \in N$. Then the slice $M \times \{p\}$ is an embedded submanifolds of $M \times N$, and it is diffeomorphic to M.*

Theorem 5.4. *Let M, N be smooth manifolds with dimension m, n respectively. Let $U \subset M$ be open and $f : U \to N$ be a smooth map. Let $\Gamma(f) \subset M \times N$ be the graph of f. Then $\Gamma(f)$ is an embedded m-dimensional submanifold of $M \times N$.*

Theorem 5.5. *Let M be a smooth manifold with or without boundary and $S \subset M$ an embedded submanifold. Then S is properly embedded if and only if it is a closed subset of M.*

Theorem 5.6. *Every compact embedded submanifold is properly embedded.*

Theorem 5.7. *Let M, N be smooth manifolds, where N may have boundary, and let $f : M \to N$ be a smooth map. Then $\Gamma(f)$ is a properly embedded submanifold of $M \times N$.*

Theorem 5.8 (Local Slice Criterion for Embedded Submanifolds)**.** *Let M be a smooth n-manifold. If $S \subset M$ is an embedded k-dimensional submanifold, then S satisfies the local k-slice condition. Conversely, if $S \subset M$ is a subset that satisfies the local k-slice condition, then S is a topological manifold with the subspace topology and has a smooth structure making it a k-dimensional embedded submanifold of M.*

Theorem 5.9. *Let M be a smooth n-manifold with boundary, and give ∂M the subspace topology. Then ∂M is a topological $(n-1)$-dimensional submanifold (without boundary), and it has a unique smooth structure such that it is a properly embedded submanifold of M.*

Theorem 5.10 (Constant-Rank Level Set Theorem)**.** *Let M, N be smooth manifolds and let $\phi : M \to N$ be a smooth map with constant rank r. Then every level set of ϕ is a properly embedded submanifold of codimension r in M.*

Theorem 5.11 (Submersion Level Set Theorem)**.** *Let M, N be smooth manifolds and $\phi : M \to N$ be a smooth submersion. Then every level set of ϕ is a properly embedded submanifold of M, with codimension equal to the dimension of N.*

Theorem 5.12. *Let M, N be smooth manifolds and $\phi : M \to N$ a smooth map. Suppose that $\dim M < \dim N$. Then every point in M is a critical point of ϕ.*

Theorem 5.13. *Let M, N be smooth manifolds and $\phi : M \to N$ a smooth submersion. Then every point in M is a regular point.*

Theorem 5.14. *Let M, N be smooth manifolds and $\phi : M \to N$ a smooth map. The sets of regular points of ϕ is an open subset of M.*

Theorem 5.15 (Regular Level Set Theorem)**.** *Every regular level set of a smooth map between smooth manifolds is a properly embedded submanifold with codimension equal to the dimension of the codomain.*

Theorem 5.16. *Let S be a subset of a smooth m-manifold M. Then S is an embedded k-submanifold of M if and only if every point of S has a neighborhood $U \subset M$ such that $U \cap S$ is a level set of a smooth submersion $\phi : U \to \mathbb{R}^{m-k}$.*

Theorem 5.17. *Every embedded submanifold is also an immersed submanifold.*

Theorem 5.18. *Let M be a smooth manifold with or without boundary, and N be a smooth manifold, and $F : M \to N$ an injective smooth immersion. Let $S = F(N)$. Then S has a unique topology and smooth structure such that it is a smooth immersed submanifold of M and such that $F : N \to S$ is a diffeomorphism.*

Theorem 5.19. *Let M be a smooth manifold with or without boundary and $S \subset M$ an immersed submanifold. If any of the following holds, then S is an embedded submanifold.*

1. *S has codimension zero in M.*

2. *The inclusion map $S \hookrightarrow M$ is proper.*

3. *S is compact.*

Theorem 5.20 (Immersed Submanifolds are Locally Embedded)**.** *Let M be a smooth manifold with or without boundary and $S \subset M$ an immersed submanifold. Then for $p \in S$, there exists a neighborhood U of p with $U \subset S$ such that U is an embedded submanifold of M.*

Theorem 5.21. *Let M, N be smooth manifolds with or without boundary, and $F : M \to N$ a smooth map, and $S \subset M$ an immersed (or embedded) submanifold. Then $F|_S : S \to N$ is smooth.*

Theorem 5.22. *Let M be a smooth manifold without boundary, and $S \subset M$ an immersed submanifold, and $F : N \to M$ a smooth map such that $f(N) \subset S$. If F is continuous as a map from $N \to S$, then $F : N \to S$ is smooth.*

Theorem 5.23. *Let M be a smooth manifold and $S \subset M$ an embedded submanifold. Then every smooth map $F : N \to M$ whose image is contained in S is also smooth as a map from N to S.*

Theorem 5.24. *Let M be a smooth manifold and $S \subset M$ an immersed submanifold, and $f \in C^\infty(S)$. Then*

1. *If S is embedded, then there exists a neighborhood U of S in M and a smooth function $\widetilde{f} \in C^\infty(U)$ such that $\widetilde{f}|_S = f$.*

2. *If S is properly embedded, then the neighborhood above can be taken to be all of M.*

Theorem 5.25. *Let M be a smooth manifold with or without bondary, and $S \subset M$ an immersed (or embedded) submanifold, and $p \in S$. Then for $v \in T_pM$, we have $v \in T_pS$ if and only if there is a smooth curve $\gamma : J \to M$ whose image is contained in S, where γ is also smooth as a map into S, such that $0 \in J, \gamma(0) = p$, and $\gamma'(0) = v$.*

Theorem 5.26. *Let M be a smooth manifold and $S \subset M$ an embedded submanifold, and $p \in S$. As a subspace of T_pM, the tangent space T_pS is precisely*

$$T_pS = \{v \in T_pM : vf = 0 \text{ whenever } f \in C^\infty(M) \text{ and } f|_s = 0\}$$

Theorem 5.27. *Let M be a smooth n-manifold with boundary, and let $p \in \partial M$ and (x^i) be smooth boundary coordinates on a neighborhood of p. Write $v \in T_pM$ as $v^i \left.\frac{\partial}{\partial x^i}\right|_p$. Then*

$$\begin{aligned} v \text{ is inward pointing} &\iff v^n > 0 \\ v \text{ is outward pointing} &\iff v^n < 0 \\ v \in T_p(\partial M) &\iff v^n = 0 \end{aligned}$$

6 Chapter 6 - Sard's theorem

Theorem 6.1. *Let $A \subset \mathbb{R}^n$ with $m(A) = 0$ and $F : A \to \mathbb{R}^n$ be a smooth map. Then* $m(F(A)) = 0$.

Theorem 6.2. *Let $F : M \to N$ be a smooth map between smooth manifolds. If $A \subset M$ has measure zero, then $F(A) \subset N$ has measure zero.*

Theorem 6.3 (Sard's Theorem)**.** *Let $F : M \to N$ be a smooth map between smooth manifolds. The set of critical values of F has measure zero.*

Theorem 6.4 (Corollary to Sard's Theorem)**.** *Let $F : M \to N$ be a smooth map between smooth manifolds. If $\dim M < \dim N$, then $F(M)$ has measure zero in N.*

7 Chapter 7 - Lie groups

Theorem 7.1. *If G is a smooth manifold with a group structure such that the map $(g, h) \mapsto gh^{-1}$ is smooth, then G is a Lie group.*

Theorem 7.2. *Every Lie group homomorphism has constant rank.*

Theorem 7.3. *A Lie group homomorphism is a Lie group isomorphism if and only if it is bijective.*

Theorem 7.4. *Let G be a Lie group and $W \subset G$ an open neighborhood of the identity. Then W generates an open subgroup of G. If W is connected, it generates a connected open subgroup of G. If G is connected, then W generates G.*

Theorem 7.5. *Let G be a Lie group and let G_0 be the connected component containing the identity. Then G_0 is a normal subgroup of G and it is the only connected open subgroup. Every connected component of G is diffeomorphic to G_0.*

Theorem 7.6. *Let $F : G \to H$ be a Lie group homomorphism. Then $\ker F$ is a properly embedded Lie subgroup of G, with codimension equal to $\operatorname{rank} F$.*

Theorem 7.7. *Let $F : G \to H$ be an injective Lie group homomorphism. Then $\operatorname{im} F$ has a unique smooth manifold structure such that it is a Lie subgroup of H and $F : G \to \operatorname{im} F$ is a Lie group isomorphism.*

Theorem 7.8 (Equivariant Rank Theorem)**.** *Let G be a Lie group and let M, N be smooth manifolds such that G acts transitively (and smoothly) on M and N. If $F : M \to N$ is a smooth equivariant map, then F has constant rank.*

8 Chapter 8 - Vector fields

Theorem 8.1 (Smoothness Criterion for Vector Fields)**.** *Let M be a smooth manifold with or without boundary and $X : M \to TM$ a rough vector field. If (U, x^i) is a smooth chart on M, then the restriction of X to U is smooth if and only if its component functions with respect to U are smooth.*

Theorem 8.2 (Extension Lemma for Vector Fields)**.** *Let M be a smooth manifold with or without boundary and let $A \subset M$ be a closed subset. Suppose X is a smooth vector field on A. If U is an open subset containing A, then there is a smooth global vector field $\widetilde{X}$ on M such that $\widetilde{X}|_A = X$ and $\operatorname{supp} \widetilde{X} \subset U$.*

Theorem 8.3. *Let M be a smooth manifold with or without boundary and let $X : M \to TM$ be a rough vector field. The following are equivalent:*

1. *X is smooth.*
2. *For every $f \in C^\infty(M)$, the function Xf is smooth.*
3. *For every open subset $U \subset M$ and every $f \in C^\infty(M)$, the function Xf is smooth on U.*

Theorem 8.4. *A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf$ for a smooth vector field $X \in \mathfrak{X}(M)$.*

Theorem 8.5. *Let $F : M \to N$ be a smooth map and $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Then X and Y are F-related if and only if for every smooth real-valued function f defined on an open subset of N we have*

$$X(f \circ F) = (Yf) \circ F$$

Theorem 8.6. *Let $F : M \to N$ be a diffeomorphism. Then for every $X \in \mathfrak{X}(M)$, there is a unique smooth vector field on N that is F-related to X.*

Theorem 8.7 (Coordinate Formula for Lie Bracket of Vector Fields)**.** *Let $X, Y \in \mathfrak{X}(M)$, and let $(U, x^i$ be a smooth chart for M. We can write $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^i \frac{\partial}{\partial x^i}$ on U. Then the coordinate expression for $[X, Y]$ is given by*

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} = (XY^j - YX^j) \frac{\partial}{\partial x^j}$$

Theorem 8.8. *Let M be a smooth manifold and (U, x^i) be local coordinates. Then*

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0$$

Theorem 8.9 (Properties of the Lie Bracket)**.** *The Lie bracket of vector fields is bilinear, antisymmetric, and satisfies the Jacobi identity. For $f, g \in C^\infty(M)$,*

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X$$

Theorem 8.10. *Let $F : M \to N$ be a smooth map, and let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ such that X_1 is F-related to Y_1 and X_2 is F-related to Y_2. Then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.*

Theorem 8.11. *Let $F : M \to N$ be a diffeomorphism and $X_1, X_2 \in \mathfrak{X}(M)$. Then*

$$F_*[X_1, X_2] = [F_* X_1, F_* X_2]$$

Theorem 8.12. *Let G be a Lie group. The set of left-invariant smooth vector fields on G is a Lie algebra under the above bracket. That is, it is a vector space and closed under brackets.*

Theorem 8.13. *Let G be a Lie group. Define $\epsilon : \mathrm{Lie}(G) \to T_eG$ by $\epsilon(X) = X_e$. Then ϵ is a vector space isomorphism. Thus $\dim \mathrm{Lie}(G) = \dim G$.*

Theorem 8.14. *Every Lie group is parallelizable.*

9 Chapter 9 - Flows

Theorem 9.1. *Let V be a smooth vector field on a smooth manifold M. For $p \in M$, there exists $\epsilon > 0$ and a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ so that γ is an integral curve of V, with $\gamma(0) = p$.*

Theorem 9.2. *Let $F : M \to N$ be a smooth map and $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Then X and Y are F-related if and only if F maps integral curves of X to integral curves of Y. That is, if γ is an integral curve of X, then $F \circ \gamma$ is an integral curve of Y.*

Theorem 9.3. *Let $\theta : \mathbb{R} \times M \to M$ be a smooth global flow on M. The infinitesimal generator of θ is a smooth vector field on M, and each curve θ^p is an integral curve of V.*

Theorem 9.4. *Let $\theta : D \to M$ be a smooth flow. The infinitesimal generator of θ is a smooth vector field and the curves θ^p are integral curves of V.*

Theorem 9.5 (Fundamental Theorem of Flows)**.** *Let V be a smooth vector field on M. There is a unique smooth maximal flow $\theta : D \to M$ whose infinitesimal generator is V. For this flow, θ^p is a maximal integral curve of V.*

Theorem 9.6. *Let $F : M \to N$ be a smooth map and $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Let θ be the flow of X and η the flow of Y. If X and Y are F-related, then the diagram commutes.*

$$\begin{array}{ccc} M_t & \xrightarrow{F} & N_t \\ {\scriptstyle \theta_t}\downarrow & & {\scriptstyle \eta_t}\downarrow \\ M_{-t} & \xrightarrow{F} & N_{-t} \end{array}$$

Theorem 9.7 (Diffeomorphism Invariance of Flows)**.** *Let $F : M \to N$ be a diffeomorphism. If $X \in \mathfrak{X}(M)$ and θ is the flow of X, then the flow of F_*X is $\eta_t = F \circ \theta_t \circ F^{-1}$.*

Theorem 9.8. *Let V be a smooth vector field on M. Suppose there exists $\epsilon > 0$ such that for every $p \in M$, the domain of θ^p contains $(-\epsilon, \epsilon)$. Then V is complete.*

Theorem 9.9. *Every compactly supported smooth vector field is complete.*

Theorem 9.10. *On a compact manifold, every smooth vector field is complete.*

Theorem 9.11. *Every left-invariant vector field on a Lie group is complete.*

Theorem 9.12. *Let V be a smooth vector field on M and let $\theta : D \to M$ be the flow of V. If $p \in M$ is a regular point, then $\theta^p : D^p \to M$ is a smooth immersion.*

Theorem 9.13. *Let M be a smooth manifold and $V, W \in \mathfrak{X}(M)$. The following are equivalent: V and W commute, V is invariant under the flow of W, and W is invariant under the flow of V.*

Theorem 9.14. *Let M be a smooth manifold, and let $V \in \mathfrak{X}(M)$. A smooth covariant tensor field A is invariant under the flow of V if and only if $L_V A = 0$.*

10 Chapter 10 - Vector bundles

Theorem 10.1. *Let E be a smooth vector bundle over M. The projection map $\pi : E \to M$ is a smooth submersion.*

Theorem 10.2. *The tangent bundle is a vector bundle.*

Theorem 10.3. *Let $\pi : E \to M$ be a smooth vector bundle of rank k over M. Suppose $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ are local trivializations such that $U \cap V \neq \emptyset$. Then there is a smooth function $\tau : U \cap V \to \mathrm{GL}(k, \mathbb{R})$ such that*

$$\phi \circ \psi^{-1}(p, v) = (p, \tau(p)v)$$

Such a map τ is called a transition function.

Theorem 10.4. *A vector field is a global section of the tangent bundle.*

11 Chapter 11 - Differential 1-forms

Theorem 11.1. *Let V be a finite-dimensional vector space. If $(E_1, \ldots, E_n)$ is a basis of V, then define $\epsilon^i : V \to \mathbb{R}$ by $\epsilon^i(E_j) = \delta_i^j$. Then $(\epsilon^1, \ldots, \epsilon^n)$ is a basis for V^*, called the dual basis.*

Theorem 11.2. *The assignment that sends a vector space to its dual space and a linear map to its dual map is a contravariant functor from the category of real vector spaces to itself.*

Theorem 11.3. *Let V be a finite-dimensional vector space and define $\xi : V \to V^{**}$ by defining $\xi(v) : V^* \to \mathbb{R}$ to be the map $\xi(v)(\omega) = \omega(v)$. Then ξ is a vector space isomorphism.*

Theorem 11.4. *If (U, x^i) is a smooth chart for a manifold, then $\left(\frac{\partial}{\partial x^i}\Big|_p \right)$ is a basis for T_pM, and its dual basis is $(dx^i|_p)$.*

Theorem 11.5. *Let $F \in C^\infty(M)$. Then $df = 0$ if and only if f is constant on each component of M.*

Theorem 11.6 (Pullback Commutes with d)**.** *Let $F : M \to N$ be a smooth map, and let $u \in C^\infty(M)$ and ω be a 1-form on N. Then*

$$F^*(du) = d(u \circ F)$$

Theorem 11.7. *Let $F : M \to N$ be a smooth map and let ω be a 1-form on N. Let (U, x^i) a smooth chart on M and (V, y^j) be a smooth chart on N with $F(U) \subset V$. Then we can write ω as $\omega = \omega_j dy^j$, and*

$$F^*\omega = (\omega_j \circ F)d(y^j \circ F) = (\omega_j \circ F)dF^j$$

12 Chapter 12 - Differential k-forms

13 Chapter 13 - Exterior derivative

14 Chapter 14 - Orientation

Theorem 14.1 (Properties of Elementary k-covectors)**.** *Let (E_i) be a basis for V and (ϵ^i) the dual basis, and I a multi-index. If I has a repeated index, $\epsilon^I = 0$. If $J = I_\sigma$ for some $\sigma \in S_k$, then $\epsilon^I = (\operatorname{sgn} \sigma)\epsilon^J$. Also,*

$$\epsilon^I(E_{j_1}, \ldots, E_{j_k}) = \delta^I_J$$

Theorem 14.2. *Let V be an n-dimensional vector space. If (ϵ^i) is a basis for V^*, then the collection*

$$\{\epsilon^I : I \text{ is an increasing multi-index of length } k\}$$

is a basis for $\Lambda^k(V^*)$.

Theorem 14.3. *Let I, J be multi-indices. Then $\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}$.*

Theorem 14.4 (Properties of Wedge Product)**.** *The wedge product is bilinear, associative, and anti-commutative. The anticommutative property says that*

$$\omega \wedge \eta = (-1)^{kl}(\eta \wedge \omega)$$

where k, l are the respective degrees of ω, η.

Theorem 14.5. *Let V be an n-dimensional vector space. If (ϵ^i) is a basis for V^*, then*

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = \epsilon^I$$

As a result,

$$\omega^1 \wedge \ldots \wedge \omega^k(v_1, \ldots, v_k) = \det(\omega^j(v_i))$$

Theorem 14.6 (Properties of Interior Multiplication)**.** *Let V be a finite-dimensional vector space and $v \in V$. Then $i_v \circ i_v = 0$, and for $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$ we have*

$$i_v(\omega \wedge \eta) = (i_v\omega) \wedge \eta + (-1)^k \omega \wedge (i_v\eta) = (v \lrcorner \omega) \wedge \eta + (-1)^k \omega \wedge (v \lrcorner \eta)$$

More generally,

$$v \lrcorner (\omega^1 \wedge \ldots \wedge \omega^k) = \sum_{i=1}^{k} (-1)^{(i-1)} \omega^i(v) \omega^1 \wedge \ldots \wedge \widehat{\omega^i} \wedge \ldots \wedge \omega^k$$

Theorem 14.7. *Let $F : M \to N$ be a smooth map. Then $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$. In a smooth chart (V, y^i) for N, we have*

$$F^*\left(\sum_I{}' \omega_I dy^{i_1} \wedge \ldots \wedge dy^{i_k}\right) = \sum_I{}' (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \ldots \wedge d(y^{i_k} \circ F)$$

Theorem 14.8 (Pullback for Top Degree Forms)**.** *Let $F : M \to N$ be smooth. Let (U, x^i) be a chart for M and (V, y^i) be a chart for N, and $f \in C^\infty(V)$. Then on $U \cap F^{-1}(V)$, we have*

$$F^*(u dy^1 \wedge \ldots \wedge dy^n) = (u \circ F)(\det J(F)) dx^1 \wedge \ldots \wedge dx^n$$

(Note that $J(F)$ is the Jacobian of F in these coordinates.)

Theorem 14.9. *If (U, x^i) and (V, y^j) are overlapping charts on M, then on $U \cap V$ we have*

$$dy^1 \wedge \ldots \wedge dy^n = \det\left(\frac{dy^j}{dx^i}\right) dx^1 \wedge \ldots \wedge dx^n$$

Theorem 14.10. *The exterior derivative exists and is unique.*

Theorem 14.11. *Let $F : M \to N$ be a smooth map. For each k, the map $F^* : \Omega^k(N) \to \Omega^k(M)$ commutes with d. That is,*

$$F^*(d\omega) = d(F^*(\omega)$$

Theorem 14.12. *Every exact form is closed.*

Theorem 14.13. *Let M be a smooth manifold and $V \in \mathcal{X}(M)$. A smooth covariant tensor field A is invariant under the flow of V if and only if $L_V A = 0$.*

Theorem 14.14 (Cartan's Magic Formula)**.** *Let M be a smooth manifold and V a smooth vector field on M, and ω a differential form on M. Then*

$$L_V \omega = V \lrcorner (d\omega) + d(V \lrcorner \omega)$$

15 Chapter 15 - Integration on manifolds

Theorem 15.1. *Let M be a smooth n-manifold. Any nonvanishing n-form on M determines a unique orientation on M. Conversely, if M is oriented, there is a smooth nonvanish n-form on M that determines that orientation.*

Theorem 15.2. *Let M be an oriented smooth manifold and $D \subset M$ a smooth codimesion-0 (immersed) submanifold. The orientation on M restricts to an orientation on D. If ω is an orientation form for M, then $\iota^*\omega$ is an orientation form for D. ($\iota : D \to M$ is the inclusion map.)*

Theorem 15.3. *Let $F : M \to N$ be a local diffeomorphism and N be oriented. Then M has a unique orientation such that F is orientation preserving.*

Theorem 15.4. *Every parallelizable manifold is orientable.*

Theorem 15.5. *Let M be an oriented smooth n-manifold with boundary. The ∂M is orientable, and all outward-pointing vector fields along ∂M determine the same orientation on ∂M.*

16 Chapter 16 - de Rham cohomology

Theorem 16.1. *Let U and V be open sets in $\mathbb{R}^n$ and let $G : U \to V$ be an orientation-preserving diffeomorphism. If ω is a compactly supported n-form on V, then*

$$\int_U G^*\omega = \int_V \omega$$

If G is orientation reversing instead of orientation preserving, then

$$\int_U G^*\omega = -\int_V \omega$$

Theorem 16.2 (Properties of Integrals)**.** *Let M, N be non-empty oriented smooth n-manifold and ω, η compactly supported n-forms on M, and let $a, b \in \mathbb{R}$. Denote M with opposite orientation by $-M$. Then*

$$\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta$$
$$\int_{-M} = -\int_M \omega$$
$$\omega \text{ is positively oriented } \implies \int_M \omega > 0$$

Theorem 16.3. *If $F : M \to N$ is an orientation preserving diffeomorphism, then*

$$\int_M \omega = \int_N F^*\omega$$

If F is orientation reversing instead, then

$$\int_M \omega = -\int_N F^*\omega$$

Theorem 16.4. *Let M be an oriented smooth n-manifold and ω a compactly supported n-form on M. Suppose $D_1, \ldots, D_k$ are open domains of integration in $\mathbb{R}^n$ and we have smooth maps $F_i : \overline{D}_i \to M$ satisfying*

1. *F_i restricts to an orientation preserving diffeomorphism from D_i to an open subset $W_i \subset M$.*
2. *$W_i \cap W_j = \emptyset$ for $i \neq j$.*
3. *$\operatorname{supp} \omega \subset \overline{W}_1 \cup \ldots \cup \overline{W}_k$*

Then

$$\int_M \omega = \sum_{i=1}^{k} \int_{D_i} F^*\omega$$

The above theorem says, in an overly complicated way, that if we parametrize our manifold M by finitely many charts that don't overlap, then we can integrate ω over M by integrating the pullback of ω on each chart. All the technicalities of the theorem say that you don't actually need to parametrize the whole manifold - you just need to parametrize the part where ω is nonzero, and you can throw away sets of measure zero, because they don't affect integration.

Theorem 16.5 (Stokes's Theorem)**.** *Let M be an oriented smooth n-manifold with boundary, and let ω be a compactly supported smooth $(n-1)$ form on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega$$

The ω on the RHS actually means $\iota^*\omega$ where $\iota : \partial M \to M$ is the inclusion. If $\partial M = \emptyset$, then the RHS is zero.

17 Chapter 17 - Poincare duality

Theorem 17.1. *Diffeomorphic smooth manifolds have isomorphic de Rhan cohomology groups.*

Theorem 17.2. *Let M be a smooth manifold, written as a disjoint union of its connected components, $M = \coprod_i M_i$. Then $H^n(M) \cong \prod_i H^n(M_i)$.*

The above theorem means that it is trivial to compute the de Rham cohomology of a disjoint union if we know the cohomologies of the pieces.

Theorem 17.3. *Let M be connected. Then $H^0(M) \cong \mathbb{R}$.*

Theorem 17.4. *Let M and N are homotopy equivalent manifolds, then $H^n(M) \cong H^n(N)$ for all n.*

Theorem 17.5. *If M is a contractible smooth manifold, then $H^n(M) = 0$ for $n \geq 1$.*

Theorem 17.6 (Poincare Lemma)**.** *If U is a star-shaped open subset of $\mathbb{R}^n$, then $H^n(U) = 0$ for $p \geq 1$.*

Theorem 17.7. *The de Rham cohomology of $\mathbb{R}^k$ is given by*

$$H^n(\mathbb{R}^k) = \begin{cases} \mathbb{R} & n = 1 \\ 0 & n \geq 1 \end{cases}$$

Theorem 17.8. *Let M be a connected smooth manifold. Then $\phi : H^1(M) \to \operatorname{Hom}(\pi_1(M), \mathbb{R})$ is well-defined and injective.*

Theorem 17.9. *If M is a connected smooth manifold with finite fundamental group, then $H^1(M) = 0$.*

Theorem 17.10. *If M is a compact connected oriented smooth n-manifold, then $H^n(M) \cong \mathbb{R}$.*

Theorem 17.11. *Let $f : M \to N$ be a smooth map between smooth manifolds. If $\dim M < \dim N$, then f is not surjective.*

Proof. Every point in M is critical, since $\operatorname{rank} df_p < \dim M < \dim T_{f(p)}N$. Thus $f(M)$ has measure zero by Sard's Theorem. Hence $f(M) \neq N$. $\square$